International Journal of Bifurcation and Chaos, Vol. 28, No. 6 (2018) 1850069 (12 pages)

DOI: 10.1142/S0218127418500694

Hopf Bifurcation of Z_2-Equivariant Generalized Liénard Systems

Yusen Wu
School of Mathematics and Statistics,
Henan University of Science and Technology,
Luoyang, Henan 471023, P. R. China
wuyusen621@126.com

Laigang Guo* and Yufu Chen†
School of Mathematical Sciences,
University of Chinese Academy of Sciences,
Beijing 100049, P. R. China
**guolaigang@126.com*
†yfchen@ucas.ac.cn

Received November 21, 2017; Revised March 29, 2018

In this paper, we consider a class of Liénard systems, described by $\ddot{x} + f(x)\dot{x} + g(x) = 0$, with Z_2 symmetry. Particular attention is given to the existence of small-amplitude limit cycles around fine foci when $g(x)$ is an odd polynomial function and $f(x)$ is an even function. Using the methods of normal form theory, we found some new and better lower bounds of the maximal number of small-amplitude limit cycles in these systems. Moreover, a complete classification of the center conditions is obtained for such systems.

Keywords: Liénard system; Z_2-equivariant vector field; normal form; limit cycle.

1. Introduction

Dynamical systems can exhibit self-sustained oscillations, known as limit cycles, which can appear in almost all areas of science and engineering. Developing limit cycle theory is not only theoretically important, but also practically significant. The study of limit cycles was initiated by Poincaré [1882], and the later development was most motivated by the well-known Hilbert's 16th problem, one of the 23 mathematical problems proposed by Hilbert [1902]. The second part of Hilbert's 16th problem considers the maximal number of limit cycles denoted by $H(n)$, and their relative locations in planar polynomial systems of degree n. Over the past century, there have been many works in this subject. However, whether $H(2) = 4$ is still an open question. In other words, the finiteness problem remains unsolved even for quadratic polynomial systems. For cubic-degree polynomial systems, there have been many studies on finding the lower bounds of the Hilbert number $H(n)$. So far, the best result for cubic systems is $H(3) \geq 13$ [Li & Liu, 2010; Li *et al.*, 2009]. This number is believed to be below the maximal number which can be obtained for generic cubic systems. Thus, some simplified versions of Hilbert's 16th problem were presented. Among these versions, many researchers have considered the generalized Liénard system

$$\ddot{x} + f(x)\dot{x} + g(x) = 0. \tag{1}$$

*Author for correspondence

Then we rewrite (1) as a differential system in the plane

$$\dot{x} = y, \quad \dot{y} = -g(x) - yf(x), \tag{2}$$

where $f(x)$ and $g(x)$ are polynomials of degrees m and n, respectively. The researchers want to find the least upper bound, denoted by $H(m,n)$, about the number of limit cycles of system (2) for all possible polynomials f and g. Let $\hat{H}(m,n)$ denote the maximum number of small amplitude limit cycles of system (2), which can be bifurcated from a focus or center. Then, we have $H(m,n) \geq \hat{H}(m,n)$. We first introduce some existing results on $\hat{H}(m,n)$ for specific values of m and n. Blows and Lloyd [1984] proved $\hat{H}(m,1) \geq [\frac{m}{2}]$ for any m. Han [1999] gave $\hat{H}(m,2) \geq [\frac{2m+1}{3}]$ for $m \geq 2$. Christopher [1999] obtained $\hat{H}(m,3) \geq 2[\frac{3m+6}{8}]$ for $2 \leq m \leq 50$. For $n = 4$, Yu and Han [2006] gave $\hat{H}(m,4) \geq m$ for $10 \leq m \leq 14$. Moreover, there are some good results on $H(m,n)$. For $2 \leq m \leq 8$, Han *et al.* [2009] obtained $H(m,4) \geq m+3$, $m = 2,3,5,6,7,8$, $H(4,4) \geq 6$. Yang *et al.* [2010] obtained $H(m,3) \geq [\frac{3m+4}{4}]$ for $3 \leq m \leq 8$. Then, Yang and Han [2011] gave $H(m,4) \geq m + 4 - [\frac{m+1}{5}]$, $3 \leq m \leq 18$. Recently, Han and Romanovski [2013] studied polynomial Liénard systems and obtained $H(m,4) \geq H(m,3) \geq 2[\frac{m-1}{4}] + [\frac{m-1}{2}]$, $m \geq 3$.

Now, we consider the following Z_2-equivariant Liénard system

$$\dot{x} = y, \quad \dot{y} = -g_{2n-1}(x) - yf_{2m}(x), \tag{3}$$

where f and g are polynomials of degrees $2m$ and $2n-1$, respectively, and satisfy

$$g_{2n-1}(-x) = -g_{2n-1}(x), \quad f_{2m}(-x) = f_{2m}(x). \tag{4}$$

Let $H^{(2)}(m,n)$ denote the maximal number of limit cycles of system (3). We know that $H^{(2)}(1,1) = 1$, $H^{(2)}(2,1) = 2$, and $H^{(2)}(m,1) \geq m$, $m \geq 3$ in [Blows & Lloyd, 1984]. From [Luo *et al.*, 1997], we have $H^{(2)}(1,2) = 3$. Yang and Han [2007] obtained $H^{(2)}(3,2) \geq 8$ and $H^{(2)}(2,2) \geq 5$. Zang *et al.* [2004] proved $H^{(2)}(2,3) \geq 4$. Xu and Li [2012] obtained that $H^{(2)}(1,3) \geq 3$, $H^{(2)}(2,3) \geq 5$, $H^{(2)}(3,3) \geq 10$, and $H^{(2)}(4,3) \geq 10$. Recently, Xiong and Zhong [2013] gave $H^{(2)}(1,3) \geq 5$, $H^{(2)}(2,3) \geq 6$, $H^{(2)}(4,3) \geq 12$.

Consider bifurcation of limit cycles associated with a singular point, then we need Lyapunov constants to determine the number and stability of bifurcating limit cycles. There mainly exist three methods for computing Lyapunov constants: the method of normal forms [Han & Yu, 2012; Farr *et al.*, 1989; Yu, 1998], the method of Poincaré return map [Andronov, 1973; Liu *et al.*, 2008], and the Lyapunov function method [Shi, 1984; Gasull & Torregrosa, 2001]. A comparative study of time-consuming task between the different methods is given in [Giné & Santallusia, 2004]. In this paper, we will compute the normal form to study bifurcation of limit cycles in the following system:

$$\begin{aligned} \frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= \delta y - (x - x^3)(b_2x^4 + b_1x^2 + b_0) \\ &\quad - y(a_4x^8 + a_3x^6 + a_2x^4 + a_1x^2 + a_0), \end{aligned} \tag{5}$$

where $a_0, a_1, a_2, a_3, a_4, b_0, b_1, b_2$ are real parameters, and $|\delta| \ll 1$. System (5) is invariant under the change $x \to -x$, $y \to -y$, in other words, system (5) is a Z_2-symmetric system. Obviously, the points $(\pm 1, 0)$ are two singular points of system (5).

The main goal of this paper is to consider the bifurcation of limit cycles around the two singular points $(\pm 1, 0)$ in the Liénard system (5). We will apply the method of normal forms to obtain a new lower bound on the number of limit cycles for $H^{(2)}(3,4)$ and $H^{(2)}(4,4)$. And a set of center conditions is obtained for such systems. In the next section, we present some basic formulations and preliminary results which are needed in proving our main results in Secs. 3 and 4. We make numerically simulations to illustrate the centers in Sec. 5. Conclusion is drawn in Sec. 6.

2. Preliminary Results

In this section, some preliminary results are presented, and will be used in the following sections. We can find the general normal form theory in [Guckenheimer & Holmes, 1983; Chow *et al.*, 1994]. Computations using computer algebra systems can be found in [Han & Yu, 2012; Tian & Yu, 2013]. Recently, an explicit recursive formula has been developed for computing the normal form together with center manifold for general n-dimensional differential systems. For brevity, we omit the detailed formulas, algorithms and the Maple program here, which can be found in [Tian & Yu, 2013].

Now, we discuss how to determine the maximal number of limit cycles bifurcating from a Hopf critical point. We first suppose that the normal form has been obtained in polar coordinates (interested readers can find the details of normal form computation in [Yu, 1998]), given by

$$\begin{aligned}\dot{r} &= r(v_0 + v_1 r^2 + v_2 r^4 + \cdots + v_k r^{2k}),\\ \dot{\theta} &= \omega_c + t_1 r^2 + t_2 r^4 + \cdots + t_k r^{2k},\end{aligned} \tag{6}$$

where r and θ denote the amplitude and phase of motion, respectively. v_k and t_k are expressed in terms of the original system's coefficients. v_k is called the kth-order focus value of the origin. The zero-order focus value v_0 is obtained from linear analysis. For finding k small-amplitude limit cycles of system (6) around the origin, we should find the conditions based on the original system's coefficients such that $v_0 = v_1 = v_2 = \cdots = v_{k-1} = 0$, but $v_k \neq 0$. Then appropriate small perturbations are found to prove the existence of k limit cycles. In the following theorem, we give sufficient conditions for the existence of small-amplitude limit cycles. (The proof can be found in [Han & Yu, 2012].)

Lemma 1. *Suppose that the focus values depend on k parameters, expressed as*

$$v_j = v_j(\epsilon_1, \epsilon_2, \ldots, \epsilon_k), \quad j = 0, 1, \ldots, k, \tag{7}$$

satisfying

$$\begin{aligned} &v_j(0, \ldots, 0) = 0, \quad j = 0, 1, \ldots, k-1,\\ &\qquad v_k(0, \ldots, 0) \neq 0 \quad \text{and}\\ &\det\left[\frac{\partial(v_0, v_1, \ldots, v_{k-1})}{\partial(\epsilon_1, \epsilon_2, \ldots, \epsilon_k)}(0, \ldots, 0)\right] \neq 0.\end{aligned} \tag{8}$$

Then, for any given $\epsilon_0 > 0$, there exist $\epsilon_1, \epsilon_2, \ldots, \epsilon_k$ and $\delta > 0$ with $|\epsilon_j| < \epsilon_0$, $j = 1, 2, \ldots, k$ such that the equation $\dot{r} = 0$ has exactly k real positive roots [i.e. system (6) has exactly k limit cycles] in a δ-ball with its center at the origin.

Next, we turn to describe the analytic conditions for centers in system (2) obtained by Cherkas [1972], also seen in [Christopher, 1999]. For system (2), assume that the singular point is at the origin $g(0) = 0$, and which is nondegenerate $g'(0) > 0$. The focal type implies that $f(0)^2 < 4g'(0)$. Moreover, we denote $F(x) = \int_0^x f(x)dx$ and $G(x) = \int_0^x g(x)dx$. By the Liénard transformation $y \mapsto y + F(x)$, system (2) becomes

$$\dot{x} = y - F(x), \quad \dot{y} = -g(x). \tag{9}$$

Since $2G(x) = g'(0)x^2 + \cdots$, the invertible analytic transformation $u = \sqrt{2G(x)}\,\mathrm{sgn}(x)$ is introduced, and its inverse is $x = x(u)$. Then, system (9) is changed to the form

$$\dot{u} = \frac{g(x(u))}{u}[y - F(x(u))], \quad \dot{y} = -g(x(u)). \tag{10}$$

Because $g(x(u))/u = \sqrt{g'(0)} + O(u)$ is analytic and nonzero in a neighborhood of the origin, we can multiply the right-hand side of (10) by $u/g(x(u))$, which gives

$$\dot{u} = y - F(x(u)), \quad \dot{y} = -u. \tag{11}$$

The system (11) has the same direction field and local qualitative behavior as (10) in the neighborhood of the origin. Note that the existence of a center cannot be changed by this scaling. We consider the power series expansion of $F(x(u)) = \sum_{i=1}^{\infty} a_i u^i$. It is shown that system (11) has a center at the origin if and only if $a_{2i+1} = 0$ for $i \geq 0$, see [Christopher, 1999]. Thus, $F(x(u)) = \phi(u^2)$ and we can establish the following results.

Lemma 2 (see [Christopher, 1999]). *System (2) has a center at the origin if and only if $F(x) = \Phi(G(x))$, for some analytic function Φ, with $\Phi(0) = 0$.*

Now we consider the function $z(x)$ defined in a neighborhood of the origin by $z(x) = x(-u(x))$. Then we give the following lemma.

Lemma 3 (see [Christopher, 1999]). *System (2) has a center at the origin if and only if there exists a function $z(x)$ satisfying $F(x) = F(z)$, $G(x) = G(z)$ with $z(0) = 0$ and $z'(0) < 0$.*

The solution $z(x)$ must correspond to a common factor between $F(x) - F(z)$ and $G(x) - G(z)$ other than $x - z$. Hence we have the following corollary for the polynomial Liénard system.

Corollary 2.1 (see [Christopher, 1999]). *If system (2) with f and g polynomials has a center at the origin, then it is necessary that the resultant of $\frac{F(x)-F(z)}{x-z}$ and $\frac{G(x)-G(z)}{x-z}$ with respect to x or z vanishes. This condition is sufficient if the common factor of the two polynomials vanishes at $x = z = 0$.*

3. Hopf Bifurcation for System $(5)|_{a_4=0}$

In this section, we study system (5) with $a_4 = 0$. To make system $(5)|_{a_4=0}$ have Hopf singular points at

$(1, 0)$ and $(-1, 0)$, we need to set $a_0 = -a_1 - a_2 - a_3$ and $b_0 = -b_1 - b_2 - \frac{1}{2}$, resulting in the following system:

$$\begin{aligned}\frac{dx}{dt} &= y,\\ \frac{dy}{dt} &= \delta y - (x - x^3)\left(b_2x^4 + b_1x^2 - b_1 - b_2 - \frac{1}{2}\right)\\ &\quad - y(a_3x^6 + a_2x^4 + a_1x^2 - a_1 - a_2 - a_3).\end{aligned} \tag{12}$$

Due to the symmetry of system (12), the focus values associated with the singular points $(1, 0)$ and $(-1, 0)$ are the same, hence we only need to consider the Hopf bifurcation at the singular point $(1, 0)$. In order to study the limit cycles bifurcation around the Hopf critical point $(1, 0)$, we need to compute its focus values. To achieve this, we use the following transformation:

$$x = x_1 + 1, \quad y = y_1,$$

to transform system (12) to the following form:

$$\begin{aligned}\frac{dx_1}{dt} &= y_1,\\ \frac{dy_1}{dt} &= \delta y_1 - x_1 + b_2x_1^7 + 7b_2x_1^6 - a_3y_1x_1^6\\ &\quad + (20b_2 + b_1)x_1^5 - 6a_3y_1x_1^5 + (30b_2 + 5b_1)x_1^4\\ &\quad - (15a_3 + a_2)y_1x_1^4 + \left(24b_2 + 8b_1 - \frac{1}{2}\right)x_1^3\\ &\quad - (20a_3 + 4a_2)y_1x_1^3 + \left(8b_2 + 4b_1 - \frac{3}{2}\right)x_1^2\\ &\quad - (15a_3 + 6a_2 + a_1)y_1x_1^2\\ &\quad - (6a_3 + 4a_2 + 2a_1)y_1x_1.\end{aligned} \tag{13}$$

Clearly, the singular point $(1, 0)$ of the system (12) corresponds to the origin of (13), which is a Hopf-type critical point. We apply the method of normal forms and the Maple program in [Tian & Yu, 2013] to system (13) to obtain the focus values.

Theorem 1. *For system (13), the first five focus values at the origin are given by*

$$\begin{aligned}v_0 &= \frac{1}{2}\delta,\\ v_1 &= -a_1b_1 - 2a_1b_2 - 2a_2b_1 - 4a_2b_2\\ &\quad - 3a_3b_1 - 6a_3b_2 + \frac{1}{4}a_1 - \frac{3}{4}a_3.\end{aligned}$$

(I) *When* $b_1 \neq -2b_2 + \frac{1}{4}$,

$$\begin{aligned}v_2 &= \frac{1}{12(4b_1 + 8b_2 - 1)}(80a_2b_1^2 + 480a_2b_1b_2\\ &\quad + 640a_2b_2^2 + 80a_3b_1^2 + 800a_3b_1b_2\\ &\quad + 1280a_3b_2^2 - 8a_2b_1 - 16a_2b_2 + 40a_3b_1\\ &\quad + 80a_3b_2 - 3a_2 - 15a_3).\end{aligned}$$

(I.a) *If* $80b_1^2 + 480b_1b_2 + 640b_2^2 - 8b_1 - 16b_2 - 3 \neq 0$, *then*

$$\begin{aligned}v_3 &= -\frac{5a_3F_1}{48(80b_1^2 + 480b_1b_2 + 640b_2^2 - 8b_1 - 16b_2 - 3)},\\ v_4 &= -\frac{a_3F_2}{1728(80b_1^2 + 480b_1b_2 + 640b_2^2 - 8b_1 - 16b_2 - 3)^3},\\ v_5 &= -\frac{a_3F_3}{995328(80b_1^2 + 480b_1b_2 + 640b_2^2 - 8b_1 - 16b_2 - 3)^5}.\end{aligned}$$

(I.b) *If* $80b_1^2 + 480b_1b_2 + 640b_2^2 - 8b_1 - 16b_2 - 3 = 0$, *then*

$$v_2 = \frac{a_3(80b_1b_2 + 160b_2^2 + 12b_1 + 24b_2 - 3)}{3(4b_1 + 8b_2 - 1)},$$

further, we have

(I.b.1) *if* $a_3 = 0$,

$$v_3 = -\frac{28a_2(-1 + 20b_2)}{625(4b_1 + 8b_2 - 1)^3}(4000b_1b_2^3 + 8000b_2^4 + 1400b_1b_2^2 + 2800b_2^3 + 280b_1b_2 + 485b_2^2 + 32b_1 + 34b_2 - 8),$$

(I.b.1.1) *if* $a_2 = 0,\ v_4 = v_5 = 0,$

(I.b.1.2) *if* $a_2 \neq 0,$

$$v_4 = -\frac{774144b_2^4(13520000b_2^4 + 2011000b_2^3 + 289550b_2^2 - 37485b_2 - 1773)}{5(5b_2+1)(100b_2^2 + 15b_2 + 4)} \neq 0,$$

(I.b.2) *if* $a_3 \neq 0,$

$$v_3 = -\frac{7}{200}a_2 - \frac{9}{40}a_3, \quad v_4 = \frac{9}{100}a_3 \neq 0.$$

(II) *When* $b_1 = -2b_2 + \frac{1}{4},$

$$v_1 = -\frac{3}{2}a_3 - \frac{1}{2}a_2, \quad v_2 = 5a_3b_2 - \frac{5}{3}a_1b_2 - \frac{1}{3}a_3.$$

(II.a) *When* $b_2 = 0,$

$$v_2 = -\frac{1}{3}a_3, \quad v_3 = v_4 = v_5 = 0.$$

(II.b) *When* $b_2 \neq 0,$

$$v_3 = -\frac{1}{12}a_3(42b_2 - 5), \quad v_4 = \frac{25}{126}a_3, \quad v_5 = 0.$$

In the above expression of v_k, *we have used* $v_1 = \cdots = v_{k-1} = 0$, *for* $k = 2, 3, 4, 5$. *Here,*

$$\begin{aligned} F_1 &= 1792b_1^4 + 7168b_1^3b_2 - 28672b_1b_2^3 - 28672b_2^4 - 1792b_1^3 - 5376b_1^2b_2 + 5376b_1b_2^2 \\ &\quad + 17920b_2^3 - 96b_1^2 - 1920b_1b_2 - 3456b_2^2 + 144b_1 + 288b_2 - 9, \\ F_2 &= \frac{226170372096}{49}b_1^3 - \frac{38690390016}{343}b_1^2 - \frac{121818193920}{343}b_1 - \frac{104806047744}{343}b_2 \\ &\quad + \frac{4044350767104}{49}b_2^4 + 1009023713280000b_2^8 - 281388515328000b_2^7 - \frac{2884805591040}{7}b_2^6 \\ &\quad - 15741789143040b_2^5 + \frac{1666892058624}{49}b_2^3 - \frac{730006401024}{343}b_2^2 + \frac{8135313408}{343} \\ &\quad - \frac{862403936256}{343}b_1b_2 + \frac{197976047616}{7}b_1b_2^2 + \frac{811348918272}{49}b_1^2b_2 + \frac{16934215974912}{49}b_1b_2^3 \\ &\quad + \frac{16273599528960}{49}b_1^2b_2^2 + \frac{4310115287040}{49}b_1^3b_2 + \frac{10693973901312}{7}b_1b_2^4 + \frac{16974222852096}{7}b_1^2b_2^3 \\ &\quad + \frac{4436211400704}{7}b_1^3b_2^2 + \frac{71600339681280}{7}b_1b_2^5 + \frac{69253276631040}{7}b_1^2b_2^4 + \frac{16365952696320}{7}b_1^3b_2^3 \\ &\quad + 1749653913600b_1b_2^6 + 180422443008000b_1^2b_2^5 + 50663522304000b_1^3b_2^4 \\ &\quad + 151353556992000b_1b_2^7 + 756767784960000b_1^2b_2^6 + 126127964160000b_1^3b_2^5 \end{aligned}$$

and F_3 *is a polynomial of* a_3, b_1 *and* b_2, *which is given in the website*: *http://math.haust.edu.cn/teacher/wuyusen.*

3.1. $H^{(2)}(3,4) \geq 10$

Theorem 2. *System (12) can have ten limit cycles with five each around the singular points* $(1,0)$ *and* $(-1,0)$. *In other words,* $H^{(2)}(3,4) \geq 10$, *that is* $H(6,7) \geq 10$.

Proof. First, note $v_0 = \frac{1}{2}\delta$. We set $\delta = 0$ to let $v_0 = 0$. To obtain the maximal number of small-amplitude limit cycles bifurcating from the origin in system (13), we first assume that

$$(4b_1 + 8b_2 - 1)(80b_1^2 + 480b_1b_2 + 640b_2^2 - 8b_1 - 16b_2 - 3) \neq 0.$$

Then, by linearly solving $V_1 = 0$ for a_1, we have

$$a_1 = -\frac{8a_2b_1 + 16a_2b_2 + 12a_3b_1 + 24a_3b_2 + 3a_3}{4b_1 + 8b_2 - 1}.$$

Further, setting $v_2 = 0$ yields that

$$a_2 = -\frac{5a_3(16b_1^2 + 160b_1b_2 + 256b_2^2 + 8b_1 + 16b_2 - 3)}{80b_1^2 + 480b_1b_2 + 640b_2^2 - 8b_1 - 16b_2 - 3}.$$

To find the solutions of $F_1 = F_2 = 0$, we use the Maple built-in command *resultant*, yielding

$$\begin{aligned} F_{12} = {} & -93361504379519888711435703099064328b_2^8 \\ & \times (10b_2 - 1)^3(20b_2 - 1)^4(6544857088b_2^7 \\ & - 49723914 24b_2^6 + 1654290176b_2^5 \\ & - 282632448b_2^4 + 22600568b_2^3 - 150356b_2^2 \\ & - 82918b_2 + 2057). \end{aligned}$$

Now, by solving $F_{12} = 0$, we can obtain six solutions for b_2, which in turn yield corresponding six solutions for b_1. However, by checking $v_3 = v_4 = 0$, we found that only three sets of them satisfy the original functions. We take one set of the solutions:

$$b_2 = 0.1627838586\cdots, \quad b_1 = -0.4330526646\cdots.$$

In addition, we choose $a_3 = 1$. Then, the other two parameters are equal to

$$a_2 = -6.6677865869\cdots, \quad a_1 = 5.2055857234\cdots.$$

The above critical values can be used to define a critical point, called p_c, for which the focus values become

$$v_1 = v_2 = v_3 = v_4 = 0, \quad v_5 = -0.5681085382\cdots.$$

Moreover, a direct calculation shows that the Jacobian evaluated at the critical point p_c is given by

$$\det\left[\frac{\partial(v_1, v_2, v_3, v_4)}{\partial(a_1, a_2, b_1, b_2)}\right] = -3.2675348817\cdots \neq 0,$$

implying, by Theorem 2.1, that system (13) can indeed have five small-amplitude limit cycles bifurcating from the center-type singular point (the origin). Thus, system (12) can have ten limit cycles.

The proof of Theorem 3.2 is complete. ■

3.2. *Center conditions in system (12)*

In this section, we derive the center condition of system (13), under which both the critical points $(1,0)$ and $(-1,0)$ of system (12) are centers. Note that the similar problem has been studied in [Giné, 2017], which gives some center conditions for systems of the form (2) with f and g of degree ≤ 6.

Theorem 3. *System (13) has a center at the origin [i.e. the critical points* $(1,0)$ *and* $(-1,0)$ *of system (12) are centers] if and only if* $\delta = 0$ *and one of the following conditions holds*:

(a): $a_1 = a_2 = a_3 = 0$,

(b): $a_2 = a_3 = 0, \quad b_2 = 0, \quad b_1 = \frac{1}{4}$.

Proof. For the case (a), system $(13)|_{\delta=0}$ becomes

$$\begin{aligned} \frac{dx_1}{dt} &= y_1, \\ \frac{dy_1}{dt} &= -x_1 + b_2x_1^7 + 7b_2x_1^6 + (b_1 + 20b_2)x_1^5 \\ &\quad + (5b_1 + 30b_2)x_1^4 + \left(8b_1 + 24b_2 - \frac{1}{2}\right)x_1^3 \\ &\quad + \left(4b_1 + 8b_2 - \frac{3}{2}\right)x_1^2, \end{aligned} \tag{14}$$

which is a Hamiltonian system with the Hamiltonian function

$$\begin{aligned} H(x_1, y_1) = {} & \frac{1}{2}y_1^2 - \frac{1}{24}x_1^2(x_1 + 2)^2 \\ & \times (3b_2x_1^4 + 12b_2x_1^3 + 4b_1x_1^2 \\ & + 20b_2x_1^2 + 8b_1x_1 + 16b_2x_1 - 3). \end{aligned}$$

The first integral is well-defined in a neighborhood of the origin, thus system (14) has a center at the origin.

For case (b), system $(13)|_{\delta=0}$ takes the form

$$\begin{aligned}\frac{dx_1}{dt} &= y_1,\\ \frac{dy_1}{dt} &= -x_1 + \frac{1}{4}x_1^5 + \frac{5}{4}x_1^4 + \frac{3}{2}x_1^3\\ &\quad - \frac{1}{2}x_1^2 - a_1y_1x_1^2 - 2a_1y_1x_1.\end{aligned} \tag{15}$$

For system (15), the primitives of $g(x_1)$ and $f(x_1)$ are

$$\begin{aligned}G(x_1) &= \frac{1}{2}x_1^2 - \frac{1}{24}x_1^6 - \frac{1}{4}x_1^5 - \frac{3}{8}x_1^4 + \frac{1}{6}x_1^3,\\ F(x_1) &= \frac{1}{3}a_1x_1^3 + a_1x_1^2.\end{aligned}$$

Moreover, we have

$$\begin{aligned}\frac{G(x_1)-G(z)}{x_1-z} &= -\frac{1}{24}(z+2+x_1)\\ &\quad \times (x_1^2 - x_1z + z^2 + x_1 + z - 2)\\ &\quad \times (x_1^2 + x_1z + z^2 + 3x_1 + 3z)\end{aligned}$$

and

$$\begin{aligned}&\frac{F(x_1)-F(z)}{x_1-z}\\ &\quad = \frac{1}{3}a_1(x_1^2 + x_1z + z^2 + 3x_1 + 3z).\end{aligned}$$

The resultant of both expressions with respect to x_1 or z is zero because both expressions have the common factor $x_1^2 + x_1z + z^2 + 3x_1 + 3z$ that vanish at $x_1 = z = 0$. Hence, by Corollary 2.1, the origin of system (15) is a center. Thus, the critical points $(1,0)$ and $(-1,0)$ of the system $(12)|_{\delta=0}$ are centers under the condition (a) or (b).

This completes the proof. ■

4. Hopf Bifurcation for System (5)

Since we have studied system (5) with $a_4 = 0$, we will assume that $a_4 \neq 0$ in system (5) in this section. Then, we can use parameter scaling and state variable scaling in (5) so that $a_4 = 1$. To make system (5) have Hopf singular points at $(1,0)$ and $(-1,0)$, we set $a_0 = -a_1 - a_2 - a_3 - 1$ and $b_0 = -b_1 - b_2 - \frac{1}{2}$, yielding the following system,

$$\begin{aligned}\frac{dx}{dt} &= y,\\ \frac{dy}{dt} &= \delta y - (x - x^3)\left(b_2x^4 + b_1x^2 - b_1 - b_2 - \frac{1}{2}\right)\\ &\quad - y(x^8 + a_3x^6 + a_2x^4 + a_1x^2\\ &\quad - a_1 - a_2 - a_3 - 1).\end{aligned} \tag{16}$$

Due to the symmetry of system (16), the focus values associated with the singular points $(1,0)$ and $(-1,0)$ are the same, hence we only need to consider the Hopf bifurcation at the singular point $(1,0)$. In order to study the limit cycles bifurcation around the Hopf critical point $(1,0)$, we need to compute its focus values. To achieve this, we use the following transformation,

$$x = x_1 + 1, \quad y = y_1,$$

to transform system (16) to the following form,

$$\begin{aligned}\frac{dx_1}{dt} &= y_1,\\ \frac{dy_1}{dt} &= \delta y_1 - x_1 + b_2x_1^7 + 7b_2x_1^6\\ &\quad + (20b_2 + b_1)x_1^5 + (5b_1 + 30b_2)x_1^4\\ &\quad - x_1^8y_1 - 8x_1^7y_1 - (a_2 + 70 + 15a_3)y_1x_1^4\\ &\quad - (4a_2 + 56 + 20a_3)y_1x_1^3 - (a_3 + 28)y_1x_1^6\\ &\quad - (56 + 6a_3)y_1x_1^5 - (15a_3 + 6a_2\\ &\quad + a_1 + 28)x_1^2y_1 - (4a_2 + 6a_3 + 2a_1 + 8)x_1y_1\\ &\quad + \left(8b_1 + 24b_2 - \frac{1}{2}\right)x_1^3 + \left(4b_1 + 8b_2 - \frac{3}{2}\right)x_1^2.\end{aligned} \tag{17}$$

Obviously, the singular point $(1,0)$ of the system (16) corresponds to the origin of (17), which is a Hopf-type critical point. Then the method of normal forms and the Maple program in [Tian & Yu, 2013] are used for system (17) to obtain the focus values.

Theorem 4. *For system (17), the first five focus values at the origin are given by*

$$\begin{aligned}v_0 &= \frac{1}{2}\delta,\\ v_1 &= -a_1b_1 - 2a_1b_2 - 2a_2b_1 - 4a_2b_2\end{aligned}$$

$$- 3a_3b_1 - 6a_3b_2 - 4b_1 - 8b_2 + \frac{1}{4}a_1 - \frac{3}{4}a_3 - 2,$$

(I) *When* $b_1 \neq -2b_2 + \frac{1}{4}$,

$$v_2 = \frac{1}{12(4b_1 + 8b_2 - 1)}(80a_2b_1^2 + 480a_2b_1b_2 + 640a_2b_2^2 + 80a_3b_1^2 + 800a_3b_1b_2 + 1280a_3b_2^2 - 160b_1^2 + 320b_1b_2 + 1280b_2^2 - 8a_2b_1 - 16a_2b_2 + 40a_3b_1 + 80a_3b_2 + 160b_1 + 320b_2 - 3a_2 - 15a_3 - 30),$$

(I.a) *if* $80b_1^2 + 480b_1b_2 + 640b_2^2 - 8b_1 - 16b_2 - 3 \neq 0$, *then*

$$v_3 = -\frac{5}{48(80b_1^2 + 480b_1b_2 + 640b_2^2 - 8b_1 - 16b_2 - 3)}$$
$$\times (1792a_3b_1^4 + 7168a_3b_1^3b_2 - 28672a_3b_1b_2^3 - 28672a_3b_2^4 + 14336b_1^4 + 100352b_1^3b_2 + 258048b_1^2b_2^2 + 286720b_1b_2^3 + 114688b_2^4 - 1792a_3b_1^3 - 5376a_3b_1^2b_2 + 5376a_3b_1b_2^2 + 17920a_3b_2^3 - 10752b_1^3 - 57344b_1^2b_2 - 89600b_1b_2^2 - 35840b_2^3 - 96a_3b_1^2 - 1920a_3b_1b_2 - 3456a_3b_2^2 - 896b_1^2 - 8960b_1b_2 - 14336b_2^2 + 144a_3b_1 + 288a_3b_2 + 672b_1 + 1344b_2 - 9a_3),$$

(I.a.1) *when* $D \neq 0$,

$$v_4 = \frac{7G_1}{48D}, \quad v_5 = \frac{7G_2}{1728D^3}, \quad v_6 = \frac{7G_3}{995328D^5},$$

(I.a.2) *when* $D = 0$,

$$v_3 = -\frac{5}{6(80b_1^2 + 480b_1b_2 + 640b_2^2 - 8b_1 - 16b_2 - 3)}(5376b_1^3b_2 + 32256b_1^2b_2^2 + 64512b_1b_2^3 + 43008b_2^4 + 448b_1^3 - 1792b_1^2b_2 - 16576b_1b_2^2 - 22400b_2^3 - 16b_1^2 + 800b_1b_2 + 1664b_2^2 - 60b_1 - 120b_2 + 9),$$

$$v_4 = -\frac{G_4}{1728(80b_1^2 + 480b_1b_2 + 640b_2^2 - 8b_1 - 16b_2 - 3)^3},$$

$$v_5 = -\frac{G_5}{995328(80b_1^2 + 480b_1b_2 + 640b_2^2 - 8b_1 - 16b_2 - 3)^5},$$

(I.b) *if* $80b_1^2 + 480b_1b_2 + 640b_2^2 - 8b_1 - 16b_2 - 3 = 0$, *then*

$$v_2 = \frac{80a_3b_1b_2 + 160a_3b_2^2 + 320b_1b_2 + 640b_2^2 + 12a_3b_1 + 24a_3b_2 + 36b_1 + 72b_2 - 3a_3 - 9}{3(4b_1 + 8b_2 - 1)},$$

$$v_3 = \frac{G_6}{1728(4b_1 + 8b_2 - 1)^3}, \quad v_4 = \frac{G_7}{622080(4b_1 + 8b_2 - 1)^5}, \quad v_5 = \frac{G_8}{59719680(4b_1 + 8b_2 - 1)^7},$$

(II) *When* $b_1 = -2b_2 + \frac{1}{4}$,

$$v_1 = -\frac{1}{2}a_2 - \frac{3}{2}a_3 - 3, \quad v_2 = \frac{40}{3}b_2 - \frac{1}{3}a_3 - \frac{5}{3}a_1b_2 + 5a_3b_2 - \frac{7}{3},$$

(II.a) *if* $b_2 = 0$, *then*

$$v_2 = -\frac{7}{3} - \frac{1}{3}a_3, \quad v_3 = v_4 = v_5 = v_6 = 0,$$

(II.b) *if* $b_2 \neq 0$, *then*

$$v_3 = \frac{5}{12}a_3 - 7b_2 + \frac{35}{12} - \frac{7}{2}a_3b_2, \quad v_4 = -\frac{7(162b_2 - 55)b_2}{6(-5 + 42b_2)}, \quad v_5 = \frac{21175}{34992}.$$

In the above expression of v_k, we have used $v_1 = v_2 = v_3 = v_4 = v_5 = 0$, for $k = 2, 3, 4, 5, 6$. Here,

$$\begin{aligned} D &= 1792b_1^4 + 7168b_1^3b_2 - 28672b_1b_2^3 - 28672b_2^4 - 1792b_1^3 - 5376b_1^2b_2 + 5376b_1b_2^2 \\ &\quad + 17920b_2^3 - 96b_1^2 - 1920b_1b_2 - 3456b_2^2 + 144b_1 + 288b_2 - 9, \\ G_1 &= 405504b_1^6 + 4775936b_1^5b_2 + 22167552b_1^4b_2^2 + 51183616b_1^3b_2^3 + 59834368b_1^2b_2^4 \\ &\quad + 30277632b_1b_2^5 + 2883584b_2^6 - 239616b_1^5 - 2183168b_1^4b_2 - 6848512b_1^3b_2^2 \\ &\quad - 7864320b_1^2b_2^3 + 32768b_1b_2^4 + 3997696b_2^5 - 6912b_1^4 - 181248b_1^3b_2 - 1225728b_1^2b_2^2 \\ &\quad - 3115008b_1b_2^3 - 2666496b_2^4 + 29952b_1^3 + 253440b_1^2b_2 + 741888b_1b_2^2 \\ &\quad + 709632b_2^3 - 6192b_1^2 - 38304b_1b_2 - 51840b_2^2 + 216b_1 + 432b_2 + 27 \end{aligned}$$

and G_2 is a polynomial of b_1, b_2, which is given at the website: http://math.haust.edu.cn/teacher/wuyusen. Note that G_k are polynomials whose form are long and complicated for $k = 3, 4, 5, 6, 7, 8$, and these polynomials are useless for studying limit cycles and center conditions in this system, thus they are omitted here.

4.1. $H^{(2)}(4, 4) \geq 12$

Theorem 5. *System (16) can have 12 limit cycles with six each around the singular points $(1, 0)$ and $(-1, 0)$. In other words, $H^{(2)}(4, 4) \geq 12$, that is $H(8, 7) \geq 12$.*

Proof. Note that $v_0 = \frac{1}{2}\delta$. We first set $\delta = 0$ to let $v_0 = 0$. In order to obtain the maximal number of small-amplitude limit cycles bifurcating from the origin in system (17), we suppose that

$$\begin{aligned} &(4b_1 + 8b_2 - 1)(80b_1^2 + 480b_1b_2 + 640b_2^2 \\ &\quad - 8b_1 - 16b_2 - 3)(1792b_1^4 + 7168b_1^3b_2 \\ &\quad - 28672b_1b_2^3 - 28672b_2^4 - 1792b_1^3 - 5376b_1^2b_2 \\ &\quad + 5376b_1b_2^2 + 17920b_2^3 - 96b_1^2 - 1920b_1b_2 \\ &\quad - 3456b_2^2 + 144b_1 + 288b_2 - 9) \neq 0. \end{aligned}$$

Then, by linearly solving $V_1 = 0$ for a_1, we have

$$a_1 = -\frac{8a_2b_1 + 16a_2b_2 + 12a_3b_1 + 24a_3b_2 + 3a_3 + 16b_1 + 32b_2 + 8}{4b_1 + 8b_2 - 1}.$$

Setting $v_2 = 0$ yields that

$$a_2 = -\frac{5(16a_3b_1^2 + 160a_3b_1b_2 + 256a_3b_2^2 + 8a_3b_1 + 16a_3b_2 - 32b_1^2 + 64b_1b_2 + 256b_2^2 - 3a_3 + 32b_1 + 64b_2 - 6)}{80b_1^2 + 480b_1b_2 + 640b_2^2 - 8b_1 - 16b_2 - 3}$$

and by solving $v_3 = 0$, we obtain

$$\begin{aligned} a_3 &= -[224(64b_1^4 + 448b_1^3b_2 + 1152b_1^2b_2^2 + 1280b_1b_2^3 + 512b_2^4 - 48b_1^3 - 256b_1^2b_2 - 400b_1b_2^2 \\ &\quad - 160b_2^3 - 4b_1^2 - 40b_1b_2 - 64b_2^2 + 3b_1 + 6b_2)]/(1792b_1^4 + 7168b_1^3b_2 - 28672b_1b_2^3 - 28672b_2^4 \\ &\quad - 1792b_1^3 - 5376b_1^2b_2 + 5376b_1b_2^2 + 17920b_2^3 - 96b_1^2 - 1920b_1b_2 - 3456b_2^2 + 144b_1 + 288b_2 - 9). \end{aligned}$$

To find the solutions of $G_1 = G_2 = 0$, we use the Maple built-in command *resultant*, yielding

$$\begin{aligned} G_{12} &= -5968978586495344043084398598305783777339213041017876141456177\backslash \\ &\quad 220496114144797834450112014187717027772947413822608330987667456b_2^{20} \\ &\quad \times (6282354244719948595 2b_2^{15} - 16814102453248484966 4b_2^{14} + 11270716965095945011 2b_2^{13} \\ &\quad + 12155968015354036224b_2^{12} - 53676948841235546112b_2^{11} + 31491405673981673472b_2^{10} \end{aligned}$$

$$- 8929402795899502592b_2^9 + 12003910399092899984b_2^8 - 16994256389449984b_2^7$$
$$- 15402767132000512b_2^6 + 22067075521979920b_2^5 - 2779227453399568b_2^4 + 33899660227737b_2^3$$
$$- 22344747425280b_2^2 + 61204426400b_2 - 569088000)(6544857088b_2^7 - 4972391424b_2^6$$
$$+ 1654290176b_2^5 - 282632448b_2^4 + 22600568b_2^3 - 150356b_2^2 - 82918b_2 + 2057)^2$$
$$\times (5488b_2^3 - 5432b_2^2 + 1351b_2 - 88)^3.$$

By solving $G_{12} = 0$, we can obtain 16 solutions for b_2, which in turn yield corresponding 16 solutions for b_1. However, by checking $v_4 = v_5 = 0$, we found that only nine sets of them satisfy the original functions. We take one set of the solutions:

$$b_2 = -0.7014257217\cdots,$$
$$b_1 = 2.2369878897\cdots.$$

Then, the other three parameters are equal to

$$a_3 = -3.2199816513\cdots,$$
$$a_2 = 2.0239319224\cdots,$$
$$a_1 = 3.0124500143\cdots.$$

The above critical values can be used to define a critical point, called p_c, for which the focus values become

$$v_1 = v_2 = v_3 = v_4 = v_5 = 0,$$
$$v_6 = 59.3476925790\cdots.$$

Moreover, a direct calculation shows that the Jacobian evaluated at the critical point p_c is given by

$$\det\left[\frac{\partial(v_1, v_2, v_3, v_4, v_5)}{\partial(a_1, a_2, a_3, b_1, b_2)}\right] = 4061.6310505035\cdots \neq 0,$$

implying, by Theorem 2.1, that system (17) can indeed have six small-amplitude limit cycles bifurcating from the center-type singular point (the origin). Thus, system (16) can have 12 limit cycles.

The proof of Theorem 4.2 is complete. ■

4.2. *Center conditions in system (16)*

In this section, we derive the center condition of system (17), under which both the critical points $(1, 0)$ and $(-1, 0)$ of system (16) are centers. Note that Giné studied the similar problem, and gave some center conditions for systems of the form (2) with f and g of degree ≤ 6 in [Giné, 2017]. By analyzing the focus values that we obtained, we have the following result.

Theorem 6. *System (17) has a center at the origin [i.e. the critical points $(1, 0)$ and $(-1, 0)$ of system (16) are centers] if and only if $\delta = 0$ and the following conditions hold:*

$$b_1 = \frac{1}{4}, \quad a_2 = 15, \quad b_2 = 0, \quad a_3 = -7.$$

Proof. Under the conditions we gave, system $(17)|_{\delta=0}$ becomes

$$\begin{aligned}\frac{dx_1}{dt} &= y_1,\\ \frac{dy_1}{dt} &= -x_1 + \frac{1}{4}x_1^5 + \frac{5}{4}x_1^4 - x_1^8 y_1\\ &\quad - 8x_1^7 y_1 + 20y_1 x_1^4 + 24y_1 x_1^3\\ &\quad - 21y_1 x_1^6 - 14y_1 x_1^5 - (13 + a_1)x_1^2 y_1\\ &\quad - (26 + 2a_1)x_1 y_1 + \frac{3}{2}x_1^3 - \frac{1}{2}x_1^2.\end{aligned} \tag{18}$$

For system (15), the primitives of $g(x_1)$ and $f(x_1)$ are

$$G(x_1) = \frac{1}{2}x_1^2 - \frac{1}{24}x_1^6 - \frac{1}{4}x_1^5 - \frac{3}{8}x_1^4 + \frac{1}{6}x_1^3,$$
$$F(x_1) = \frac{1}{9}x_1^9 + x_1^8 - 4x_1^5 - 6x_1^4 + 3x_1^7 + \frac{7}{3}x_1^6 + \frac{1}{3}(13 + a_1)x_1^3 + (13 + a_1)x_1^2.$$

Moreover, we have

$$\frac{G(x_1) - G(z)}{x_1 - z} = -\frac{1}{24}(z + 2 + x_1) \times (x_1^2 - x_1 z + z^2 + x_1 + z - 2) \times (x_1^2 + x_1 z + z^2 + 3x_1 + 3z)$$

and

$$\begin{aligned}\frac{F(x_1)-F(z)}{x_1-z} &= \frac{1}{9}(x_1^2+x_1z+z^2+3x_1+3z)\\ &\quad\times(x_1^6+x_1^3z^3+z^6+6x_1^5\\ &\quad+3x_1^3z^2+3x_1^2z^3+6z^5+9x_1^4\\ &\quad+9x_1^2z^2+9z^4-6x_1^3-6z^3\\ &\quad-18x_1^2-18z^2+3a_1+39).\end{aligned}$$

The resultant of both expressions with respect to x_1 or z is zero because both expressions have the common factor $x_1^2+x_1z+z^2+3x_1+3z$ that vanish at $x_1=z=0$. Hence, by Corollary 2.1, the origin of system (18) is a center. Thus, the critical points $(1,0)$ and $(-1,0)$ of the system $(16)|_{\delta=0}$ are centers under the conditions.

This completes the proof. ■

5. Conclusion

In this paper, we have given two new lower bounds of the number of small-amplitude limit cycles around two critical points, i.e. $H^{(2)}(3,4) \geq 10$ and $H^{(2)}(4,4) \geq 12$. Normal form theory has been applied to compute the focus values, and then to determine the number of bifurcating limit cycles near the critical points. Moreover, based on the normal forms, two sets of center conditions for the two critical points have been obtained for such two kinds of systems, respectively.

Acknowledgments

This work is supported in part by the National Natural Science Foundation of China (Nos. 11101126 and 11271363), Scientific Research Foundation for Doctoral Scholars of Henan University of Science and Technology (No. 13480021) and the Foundation for Youth Leaders of Disciplines in Science of Henan University of Science and Technology (No. 13490003). L. Guo thanks the University of Chinese Academy of Sciences Joint PhD Training Program for the support received.

References

Andronov, A. A. [1973] *Theory of Bifurcations of Dynamic Systems on a Plane* (Wiley, NY).

Blows, T. R. & Lloyd, N. G. [1984] "The number of small-amplitude limit cycles of Liénard equations," *Math. Proc. Camb. Philos. Soc.* **95**, 359–366.

Cherkas, L. A. [1972] "On the conditions for a center for certain equations of the form $yy' = P(x) + Q(x)y + R(x)y^2$," *Diff. Eqs.* **8**, 1104–1107.

Chow, S. N., Li, C. & Wang, D. [1994] *Normal Forms and Bifurcation of Planar Vector Fields* (Cambridge University Press, Cambridge).

Christopher, C. [1999] "An algebraic approach to the classification of centers in polynomial Liénard systems," *J. Math. Anal. Appl.* **229**, 319–329.

Farr, W. W., Li, C., Labouriau, I. S. & Langford, W. F. [1989] "Degenerate Hopf bifurcation formulas and Hilbert's 16th problem," *SIAM J. Math. Anal.* **20**, 13–30.

Gasull, A. & Torregrosa, J. [2001] "A new approach to the computation of the Lyapunov constants," *Comput. Appl. Math.* **20**, 149–177.

Giné, J. & Santallusia, X. [2004] "Implementation of a new algorithm of computation of the Poincaré–Liapunov constants," *J. Comput. Appl. Math.* **166**, 465–476.

Giné, J. [2017] "Center conditions for polynomial Liénard systems," *Qual. Th. Dyn. Syst.* **16**, 119–126.

Guckenheimer, J. & Holmes, P. [1983] *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields* (Springer-Verlag, NY).

Han, M. [1999] "Liapunov constants and Hopf cyclicity of Liénard systems," *Ann. Diff. Eqs.* **15**, 113–126.

Han, M., Yan, H., Yang, J. & Lhotka, C. [2009] "On the number of limit cycles of some Liénard systems," *Can. Appl. Math. Quart.* **17**, 61–83.

Han, M. & Yu, P. [2012] *Normal Forms, Melnikov Functions and Bifurcations of Limit Cycles* (Springer-Verlag, NY).

Han, M. & Romanovski, V. [2013] "On the number of limit cycles of polynomial Liénard systems," *Nonlin. Anal.: Real World Appl.* **14**, 1655–1668.

Hilbert, D. [1902] "Mathematical problems," *Bull. Am. Math. Soc.* **8**, 437–479.

Li, C., Liu, C. & Yang, J. [2009] "A cubic system with thirteen limit cycles," *J. Diff. Eqs.* **246**, 3609–3619.

Li, J. & Liu, Y. [2010] "New results on the study of Z_q-equivariant planar polynomial vector fields," *Qual. Th. Dyn. Syst.* **9**, 167–219.

Liu, Y., Li, J. & Huang, W. [2008] *Singular Point Values, Center Problem and Bifurcations of Limit Cycles of Two Dimensional Differential Autonomous Systems* (Science Press, Beijing).

Luo, D., Wang, X., Zhu, D. & Han, M. [1997] *Bifurcation Theory and Methods of Dynamical Systems*, Advanced Series in Dynamical Systems (World Scientific, Singapore).

Poincaré, H. [1882] "Mémoire sur les courbes définies par une équation différentielle (II)," *J. Math. Pures Appl.* **8**, 251–296.

Shi, S. [1984] "On the structure of Poincaré–Lyapunov constants for the weak focus of polynomial vector fields," *J. Diff. Eqs.* **52**, 52–57.

Tian, Y. & Yu, P. [2013] "An explicit recursive formula for computing the normal form and center manifold of general n-dimensional differential systems associated with Hopf bifurcation," *Int. J. Bifurcation and Chaos* **23**, 1350104-1–18.

Xiong, Y. & Zhong, H. [2013] "The number of limit cycles in a Z_2-equivariant Liénard system," *Int. J. Bifurcation and Chaos* **23**, 1350085-1–17.

Xu, W. & Li, C. [2012] "Limit cycles of some polynomial Liénard systems," *J. Math. Anal. Appl.* **389**, 367–378.

Yang, J. & Han, M. [2007] "Limit cycles near a double homoclinic loop," *Ann. Diff. Eqs.* **23**, 536–545.

Yang, J., Han, M. & Romanovski, V. [2010] "Limit cycle bifurcations of some Liénard systems," *J. Math. Anal. Appl.* **366**, 242–255.

Yang, J. & Han, M. [2011] "Limit cycle bifurcations of some Liénard systems with a cuspidal loop and a homoclinic loop," *Chaos Solit. Fract.* **44**, 269–289.

Yu, P. [1998] "Computation of normal forms via a perturbation technique," *J. Sound Vibr.* **211**, 19–38.

Yu, P. & Han, M. [2006] "Limit cycles in generalized Liénard systems," *Chaos Solit. Fract.* **30**, 1048–1068.

Zang, H., Zhang, T. & Chen, W. [2004] "Bifurcations of limit cycles from quintic Hamiltonian systems with a double figure eight loop," *J. Complexity* **20**, 544–560.